On massive gravity and bigravity in three dimensions

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Abstract

In this paper we investigate possible consistent ghost-free models containing massive spin 2 particles in three dimensions. We work in a constructive approach based on the frame-like gauge invariant description for such massive spin 2 particles. We provide the most general form of linear approximations, i.e. cubic vertices in the Lagrangian and linear in fields corrections to gauge transformations. As for the possibility to go beyond the linear approximation, we show that there exists at least one solution that admits non-singular massless limit and that corresponds to a so called "New massive gravity".

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Introduction

Constructing consistent interacting theories containing massive spin 2 particles is an old, interesting and important physical problem. One of the main difficulties one faces in such theories is the appearance of non-physical ghost degree of freedom [1]. During last three years essential progress has been achieved in this direction.

In three dimensions a so called "New massive gravity" appeared [2, 3]. It was constructed as a particular example of higher derivatives gravity but it turns out to be equivalent to the system of massless and massive spin 2 particles, the massless one being a ghost. To a great extent such construction is specific namely to spin 2 in three dimensions so that it is not an easy and straightforward task to find its generalizations to higher spins (see e.g. [4, 5]) or higher dimensions (e.g. [6, 7, 8, 9]). One of the open questions is the so called partially massless limit which exists for the free massive spin 2 in de Sitter space and where additional local gauge symmetry arises [10].

More recently a whole family of consistent ghost-free models in four dimensions has been constructed both the massive gravity [11, 12], as well as for massive bigravity [13, 14]. In general such model consist of usual action for one or two massless gravitons (non interacting in the massless limit) and complicated non-linear potential terms without derivatives. In this, there is no any particular symmetry that can guarantee and/or explain the absence of ghost degree of freedom, so to check that one has to go through careful Hamiltonian analysis [15, 16]. Even in the so called Stueckelberg formulation [17] where gauge symmetries of massless theory are restored, to check the absence of ghost one still have to use Hamiltonian analyses [18]. As in the three dimensional case, it is not at all clear what happens in such theories in the partially massless limit.

In both cases it seems that it would be instructive if we can reproduce such theories in a constructive approach based on the gauge invariant description for massive higher spin particles [19, 20]. Such formalism has enough gauge symmetries to guarantee (and explain) the absence of ghosts without using careful Hamiltonian analysis. Also it seems natural to work in a frame-like formalism where the structure of potential terms becomes much more simple and clear [21]. In this paper we begin such a program starting with the $d = 3$ case.

The plan of the paper is simple. Section 1 devoted to the massless case. First of all we briefly remind a frame-like description of $d = 3$ massless gravity (just to set notations and conventions) and then consider the most general interacting theory for two massless ones. Main Section 2 devoted to the case where one of spin 2 particles is massive, while the other one remains massless. In Subsection 2.1 we give frame-like gauge invariant description of massive spin 2 particles [19] adopted to $d = 3$ case. The in Subsections 2.2 and 2.3 we consider in linear approximation self-interaction for massive spin 2 and its interaction with massless graviton, respectively. At last, in Subsection 2.4 we discuss possibilities to go beyond linear approximation. we show that if we are looking for the theory admitting non-singular massless limit that reduce to massless bi-gravity considered in Section 1, then there exists at least one solution which requires that massless graviton be a ghost exactly as in New massive gravity.

1 Massless case

In this section we consider massless gravity and bigravity as starting point for models where one of the spin 2 particles becomes massive while the other one remains massless.

1.1 Gravity

Usually frame-like formalism for gravity involves pair of fields — frame $h_\mu{}^a$ and Lorentz connection $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$. But in three dimensions it is very convenient to use dual variable $\omega_\mu{}^{ab} \to \omega_\mu{}^a = \varepsilon^{abc}\omega_\mu{}^{bc}$. In these notations the free Lagrangian describing massless spin 2 particles in $(A)dS_3$ space can be written as follows (parameter $\sigma = \pm 1$ takes into account that in $d = 3$ massless spin 2 may be a ghost):

$$\sigma \mathcal{L}_0 = \frac{1}{2} \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a \omega_\mu{}^b - \varepsilon^{\mu\nu\alpha} \omega_\mu{}^a D_\nu h_\alpha{}^a - \frac{\Lambda}{2} \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \tag{1}$$

Here Λ — cosmological constant, $\left\{ {}^{\mu\nu}_{ab} \right\} = e^\mu{}_a e^\nu{}_b - e^\mu{}_b e^\nu{}_a$ and so on, where $e_\mu{}^a$ — non-dynamical background frame while AdS_3 covariant derivatives D_μ are normalized so that

$$[D_\mu, D_\nu]\xi^a = -\Lambda e_{[\mu}{}^a \xi_{\nu]}$$

This Lagrangian is invariant under the following local gauge transformations:

$$\delta_0 h_\mu{}^a = D_\mu \hat{\xi}^a + \varepsilon_\mu{}^{ab} \hat{\eta}^b, \qquad \delta_0 \omega_\mu{}^a = D_\mu \hat{\eta}^a - \Lambda \varepsilon_\mu{}^{ab} \hat{\xi}^b \tag{2}$$

where $\hat{\eta}^a$ — dual to Lorentz transformation parameter $\hat{\eta}^a = \varepsilon^{abc}\hat{\eta}^{bc}$.

In a frame-like formalism it is easy to introduce self-interaction for such massless spin 2 particles in the linear approximation[1]. Cubic vertex has the form

$$\mathcal{L}_1 = \kappa_0 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [h_\mu{}^a \omega_\nu{}^b \omega_\alpha{}^c - \frac{\Lambda}{3} h_\mu{}^a h_\nu{}^b h_\alpha{}^c] \tag{3}$$

where κ_0 — coupling constant while corresponding corrections to gauge transformations look as follows:

$$\begin{aligned} \delta_1 h_\mu{}^a &= -2\sigma\kappa_0 \varepsilon^{abc} [h_\mu{}^b \hat{\eta}^c + \omega_\mu{}^b \hat{\xi}^c] \\ \delta_1 \omega_\mu{}^a &= -2\sigma\kappa_0 \varepsilon^{abc} [\omega_\mu{}^b \hat{\eta}^c - \Lambda h_\mu{}^b \hat{\xi}^c] \end{aligned} \tag{4}$$

A remarkable feature of $d = 3$ frame-like formalism is that there are no any quartic vertices for spin 2 (and all spins higher than 2) particles. Thus for the theory to be closed we must have $\delta_1 \mathcal{L}_1 = 0$. For the case at hands it is easy to check that these variations indeed cancel.

[1]Here and in what follows linear approximation means cubic vertices in the Lagrangian and linear in fields corrections to gauge transformations, hence the name.

1.2 Bigravity

Recall that in general $d \geq 3$ dimension there are only two possible cases for interacting theories with two massless spin 2 particles (see e.g. [22, 23, 24]). In the first one, where both spin 2 particles are physical, any interacting Lagrangian by field redefinitions can be reduced to the sum of two independent halves. In the second one we do have non-trivial cross-interaction with the price that one of the spin 2 particles must be a ghost so that such case is of interest in $d = 3$ only. Let us see how these results come in $d = 3$ frame-like formalism.

We will use the following notations for second spin 2 particle and its gauge parameters: $\Omega_\mu{}^a$, $f_\mu{}^a$, η^a and ξ^a. Let us consider interactions in the linear approximation. There are four possible cubic vertices which we denote hhh, hhf, hff and fff correspondingly. In the linear approximation they are completely independent from each other so we can consider them separately.

Vertex hhh is the same as in the previous subsection.

Vertex hhf Here cubic vertex has the form

$$\mathcal{L}_1 = \kappa_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [f_\mu{}^a \omega_\nu{}^b \omega_\alpha{}^c + 2h_\mu{}^a \omega_\mu{}^b \Omega_\alpha{}^c - \frac{\Lambda}{2} f_\mu{}^a h_\nu{}^b h_\alpha{}^c] \tag{5}$$

while corrections to gauge transformations look like:

$$\begin{aligned} \delta_1 \omega_\mu{}^a &= -2\kappa_1 \varepsilon^{abc} [\Omega_\mu{}^b \hat{\eta}^c + \omega_\mu{}^b \eta^c - \Lambda f_\mu{}^b \hat{\xi}^c - \Lambda h_\mu{}^b \xi^c] \\ \delta_1 h_\mu{}^a &= -2\kappa_1 \varepsilon^{abc} [f_\mu{}^b \hat{\eta}^c + h_\mu{}^b \eta^c + \Omega_\mu{}^b \hat{\xi}^c + \omega_\mu{}^b \xi^c] \\ \delta_1 \Omega_\mu{}^a &= -2\kappa_1 \varepsilon^{abc} [\omega_\mu{}^b \hat{\eta}^c - \Lambda h_\mu{}^b \hat{\xi}^c] \\ \delta_1 f_\mu{}^a &= -2\kappa_1 \varepsilon^{abc} [h_\mu{}^b \hat{\eta}^c + \omega_\mu{}^b \hat{\xi}^c] \end{aligned} \tag{6}$$

Vertex hff This case is similar to the previous one but roles of two fields are interchanged:

$$\mathcal{L}_1 = \kappa_2 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [h_\mu{}^a \Omega_\nu{}^b \Omega_\alpha{}^c + 2f_\mu{}^a \omega_\nu{}^b \Omega_\alpha{}^c - \frac{\Lambda}{2} h_\mu{}^a f_\nu{}^b f_\alpha{}^c] \tag{7}$$

$$\begin{aligned} \delta_1 \omega_\mu{}^a &= -2\kappa_2 \varepsilon^{abc} [\Omega_\mu{}^b \eta^c - \Lambda f_\mu{}^b \xi^c] \\ \delta_1 h_\mu{}^a &= -2\kappa_2 \varepsilon^{abc} [f_\mu{}^b \eta^c + \Omega_\mu{}^b \xi^c] \\ \delta_1 \Omega_\mu{}^a &= -2\kappa_2 \varepsilon^{abc} [\Omega_\mu{}^b \hat{\eta}^c + \omega_\mu{}^b \eta^c - \Lambda f_\mu{}^b \hat{\xi}^c - \Lambda h_\mu{}^b \xi^c] \\ \delta_1 f_\mu{}^a &= -2\kappa_2 \varepsilon^{abc} [f_\mu{}^b \hat{\eta}^c + h_\mu{}^b \eta^c + \Omega_\mu{}^b \hat{\xi}^c + \omega_\mu{}^b \eta^c] \end{aligned} \tag{8}$$

Vertex fff And this case is similar to hhh one:

$$\mathcal{L}_1 = \kappa_3 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [f_\mu{}^a \Omega_\nu{}^b \Omega_\alpha{}^c - \frac{\Lambda}{3} f_\mu{}^a f_\nu{}^b f_\alpha{}^c] \tag{9}$$

$$\begin{aligned} \delta_1 \Omega_\mu{}^a &= -2\kappa_3 \varepsilon^{abc} [\Omega_\mu{}^b \eta^c - \Lambda f_\mu{}^b \xi^c] \\ \delta_1 f_\mu{}^a &= -2\kappa_3 \varepsilon^{abc} [f_\mu{}^b \eta^c + \Omega_\mu{}^b \xi^c] \end{aligned} \tag{10}$$

Note that as can be seen from formulas given above gauge transformations mix two our spin 2 fields so that if we try to go beyond linear approximation these four cubic vertices

will not be independent any more. And here we again face the fact that in $d = 3$ frame-like formalism there are no any quartic vertices so that all variations of cubic ones must cancel each other. Happily, this is indeed possible provided the following relation holds:

$$\kappa_1{}^2 + \kappa_2{}^2 - \sigma\kappa_0\kappa_2 - \kappa_1\kappa_3 = 0 \qquad (11)$$

Thus we have solution with three parameters and their meaning is rather clear: we have two spin 2 particles and thus two independent coupling constants and also a kind of "mixing angle". This last parameter is related with the fact that we have two similar particles and so we can make field redefinition mixing them. But in the case where one of the particles become massive while the other one remains massless this symmetry between them is broken so we will not try to make such redefinition[2]. Instead, we will use the fact that there are severe restrictions on the possible cubic vertices with two massless and one massive spin particles. In general $d \geq 4$ case [25, 26] such vertex requires as many as 6 derivatives and in $d = 3$ it is just absent (see Appendix B). Thus, assuming that it is the second particle $(f_\mu{}^a,\ \Omega_\mu{}^a)$ that will become massive, we must set $\kappa_1 = 0$, in this from the relation (11) we immediately obtain that[3] $\kappa_2 = \sigma\kappa_0$ while κ_3 remains arbitrary.

2 Massive case

In this section we consider models combining massless spin 2 particle (that may be physical one or ghost) and massive one. We will work in a constructive approach using frame-like gauge invariant description for massive spin 2 particles. General $d \geq 3$ case has been constructed in [19] (see also [20]) and here we give version adopted to $d = 3$ dimensions.

2.1 Gauge invariant frame-like formalism

For the description of massive spin 2 particle in $(A)dS_3$ we will use the following set of fields: $(\Omega_\mu{}^a,\ f_\mu{}^a)$, $(B^a,\ A_\mu)$ and $(\pi^a,\ \varphi)$, where $B^a = \varepsilon^{abc}F^{bc}$. Then the free Lagrangian has the form:

$$\begin{aligned}
\mathcal{L}_0 \;=\; & \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \Omega_\mu{}^a\Omega_\nu{}^b - \varepsilon^{\mu\nu\alpha}\Omega_\mu{}^a D_\nu f_\alpha{}^a + \frac{1}{2}B_a{}^2 - \varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha - \frac{1}{2}\pi_a{}^2 + \pi^\mu D_\mu\varphi + \\
& + m\varepsilon^{\mu\nu\alpha}[-2\Omega_{\mu\nu}A_\alpha + B_\mu f_{\nu\alpha}] + 2M\pi^\mu A_\mu + \\
& + \frac{M^2}{2}\left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b + 2mMe^\mu{}_a f_\mu{}^a\varphi + 3m^2\varphi^2
\end{aligned} \qquad (12)$$

where $M^2 = 2m^2 - \Lambda$. This Lagrangian is invariant under the following local gauge transformations:

$$\begin{aligned}
\delta_0\Omega_\mu{}^a &= D_\mu\eta^a + M^2\varepsilon_\mu{}^{ab}\xi^b \\
\delta_0 f_\mu{}^a &= D_\mu\xi^a + \varepsilon_\mu{}^{ab}\eta^b + 2me_\mu{}^a\xi
\end{aligned} \qquad (13)$$

[2] Clearly, we still can do such field redefinition but as a result mass terms will not be diagonal any more.

[3] This relation is nothing but usual manifestation of universality of gravitational interactions, i.e. the same coupling constant determines both self-interaction for graviton as well as its interaction with matter with massive spin 2 playing the role of matter here.

$$\begin{aligned} \delta_0 B^a &= -2m\eta^a, \qquad \delta_0 A_\mu = D_\mu\xi + m\xi_\mu \\ \delta_0 \pi^a &= 2mM\xi^a, \qquad \delta_0\varphi = -2M\xi \end{aligned}$$

Recall that in dS space ($\Lambda > 0$) there exists a so called partially massless limit $M \to 0$, where scalar field completely decouples, leaving us with the Lagrangian

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \Omega_\mu{}^a \Omega_\nu{}^b - \varepsilon^{\mu\nu\alpha}\Omega_\mu{}^a D_\nu f_\alpha{}^a + \frac{1}{2}B_a{}^2 - \varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha + \\ &+ m\varepsilon^{\mu\nu\alpha}[-2\Omega_{\mu\nu}A_\alpha + B_\mu f_{\nu\alpha}] \end{aligned} \tag{14}$$

which is still invariant under all three gauge transformations

$$\begin{aligned} \delta_0 \Omega_\mu{}^a &= D_\mu \eta^a, \qquad \delta_0 f_\mu{}^a = D_\mu \xi^a + \varepsilon_\mu{}^{ab}\eta^b + 2me_\mu{}^a\xi \\ \delta_0 B^a &= -2m\eta^a, \qquad \delta_0 A_\mu = D_\mu\xi + m\xi_\mu \end{aligned} \tag{15}$$

As a result we obtain system with only one physical degree of freedom instead of two in general massive case.

2.2 Self-interaction

In this subsection we consider possible self-interaction for massive spin 2 particle. As we have already mentioned we will work in a constructive approach where one construct the most general terms for the Lagrangian and corrections to gauge transformations and requires that the whole Lagrangian will be gauge invariant. In massive case due to large number of fields such calculations turn out to be rather complicated thus it is important to group different variations in some convenient way. In a metric-like formalism (see e.g. [27]) one may use grouping by the number of derivatives, while in a frame-like formalism where both physical and auxiliary fields are present it is convenient to group them by the mass order of coefficients. Thus for the free Lagrangian we will have $\mathcal{L}_0 = \mathcal{L}_{00} + \mathcal{L}_{01} + \mathcal{L}_{02}$ where $\mathcal{L}_{00}$ — kinetic terms while $\mathcal{L}_{01}$ and $\mathcal{L}_{02}$ contains terms of order m and m^2 respectively. Similarly in the linear approximation we will write cubic vertices and linear corrections to gauge transformations as

$$\mathcal{L}_1 = \mathcal{L}_{10} + \mathcal{L}_{11} + \mathcal{L}_{12}, \qquad \delta_1 = \delta_{10} + \delta_{11} + \delta_{12} \tag{16}$$

This implies that we begin with some massless theory satisfying

$$\delta_{00}\mathcal{L}_{10} + \delta_{10}\mathcal{L}_{00} = 0$$

and then we proceed with the deformation of such theory to non-zero mass considering variations of order m:

$$\delta_{00}\mathcal{L}_{11} + \delta_{01}\mathcal{L}_{10} + \delta_{10}\mathcal{L}_{01} + \delta_{11}\mathcal{L}_{00} = 0$$

and so on.

Let us begin with $\mathcal{L}_{10}$. In Appendix A we show that all possible terms containing two spin 2 and one spin 0 particles can be removed by appropriate fields redefinitions. Taking into account the absence of such $2-2-0$ vertex the most general form can be written as follows:

$$\begin{aligned} \mathcal{L}_{10} &= \kappa_3 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \Omega_\mu{}^a \Omega_\nu{}^b f_\alpha{}^c + a_1 f B^a B^a + a_2 \varepsilon^{\mu\nu\alpha} f_\mu{}^a B^a D_\nu A_\alpha + a_3 \varphi B^a B^a + \\ &+ a_4 \varphi \varepsilon^{\mu\nu\alpha} B_\mu D_\nu A_\alpha + a_5 f \pi^a \pi^a + a_6 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a D_\nu \varphi \pi^b \end{aligned} \tag{17}$$

Note that possible terms of the form $\varphi\pi^2$ and $\varphi\pi D\varphi$ can also be removed by field redefinitions

$$\pi^a \Rightarrow \pi^a + \kappa_1\varphi\pi^a, \qquad \varphi \Rightarrow \varphi + \kappa_2\varphi^2$$

There is one more possible redefinition

$$B^a \Rightarrow B^a + \kappa_0\varphi B^a \tag{18}$$

that we will use later on. Let us consider variations of order m^0.
η^a **transformations:**

$$\begin{aligned}
&2\kappa_3 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [D_\mu\Omega_\nu{}^a\eta^b f_\alpha{}^c - \Omega_\mu{}^a\eta^b D_\nu f_\alpha{}^c] + 2a_0\varepsilon^{\mu\nu\alpha}\Omega_{\mu,\nu}\Omega_\alpha{}^a\eta^a + \\
&+a_2 \left\{ {}^{\mu\nu}_{ab} \right\} B^a\eta^b D_\mu A_\nu - a_6\varepsilon^{\mu\nu\alpha}\pi_\mu D_\nu\varphi\eta_\alpha
\end{aligned}$$

To compensate these variations we introduce the following corrections to gauge transformations[4]:

$$\begin{aligned}
\delta f_\mu{}^a &= -2\kappa_3\varepsilon^{abc}f_\mu{}^b\eta^c, \qquad \delta\Omega_\mu{}^a = -2\kappa_3\varepsilon^{abc}\Omega_\mu{}^b\eta^c \\
\delta B^a &= \alpha_1\varepsilon^{abc}B^b\eta^c, \qquad \delta\pi^a = \alpha_2\varepsilon^{abc}\pi^b\eta^c
\end{aligned}$$

This gives $a_2 = \alpha_1$ and $a_6 = -\alpha_2$.
ξ^a **transformations:**

$$\begin{aligned}
&-2\kappa_3 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} D_\mu\Omega_\nu{}^a\Omega_\alpha{}^b\xi^c - 2a_1\xi^\mu B^a D_\mu B^a - a_2\varepsilon^{\mu\nu\alpha}\xi^a D_\mu B^a D_\nu A_\alpha - \\
&-2a_5\xi^\mu\pi^a D_\mu\pi^a + a_6 \left\{ {}^{\mu\nu}_{ab} \right\} D_\mu\pi^a D_\nu\varphi\xi^b
\end{aligned}$$

Thus we need the following corrections:

$$\begin{aligned}
\delta f_\mu{}^a &= -2\kappa_3\varepsilon^{abc}\Omega_\mu{}^b\xi^c, \qquad \delta A_\mu = \alpha_3\varepsilon_\mu{}^{ab}B^a\xi^b, \qquad \delta B_\mu = \alpha_4\xi^a D_\mu B^a \\
\delta\varphi &= \alpha_5(\pi\xi), \qquad \delta\pi^a = \alpha_6(\xi^\mu D_\mu\pi^a - \xi^a(D\pi))
\end{aligned}$$

In this, all variations can be cancelled provided

$$2a_1 = \alpha_3 = \alpha_4 = -\alpha_1, \qquad 2a_5 = -\alpha_5 = -\alpha_6 = \alpha_2$$

Thus in this order we obtain:

$$\begin{aligned}
\mathcal{L}_{10} &= \kappa_3 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \Omega_\mu{}^a\Omega_\nu{}^b f_\alpha{}^c - \frac{\alpha_1}{2} f B^a B^a + \alpha_1\varepsilon^{\mu\nu\alpha} f_\mu{}^a B^a D_\nu A_\alpha + \\
&+a_3\varphi B^a B^a + a_4\varphi\varepsilon^{\mu\nu\alpha}B_\mu D_\nu A_\alpha + \frac{\alpha_2}{2} f\pi^a\pi^a - \alpha_2 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a D_\nu\varphi\pi^b
\end{aligned} \tag{19}$$

$$\begin{aligned}
\delta_{10}\Omega_\mu{}^a &= -2\kappa_3\varepsilon^{abc}\Omega_\mu{}^b\eta^c, \qquad \delta_{10}f_\mu{}^a = -2\kappa_3\varepsilon^{abc}[f_\mu{}^b\eta^c + \Omega_\mu{}^b\xi^c] \\
\delta_{10}B_\mu &= \alpha_1\varepsilon_\mu{}^{ab}B^a\eta^b - \alpha_1\xi^a D_\mu B^a, \qquad \delta_{10}A_\mu = -\alpha_1\varepsilon_\mu{}^{ab}B^a\xi^b \\
\delta_{10}\pi^a &= \alpha_2\varepsilon^{abc}\pi^b\eta^c - \alpha_2(\xi^\mu D_\mu\pi^a - \xi^a(D\pi)), \qquad \delta_{10}\varphi = -\alpha_2(\pi\xi)
\end{aligned} \tag{20}$$

[4]In a frame-like formalism the structure of such corrections is completely determined by the terms in variations containing explicit derivatives.

Let us consider variations of order m. The most general terms in the Lagrangian have the form:

$$
\begin{aligned}
\mathcal{L}_{11} &= \varepsilon^{\mu\nu\alpha}[b_1 f_\mu{}^a \Omega_\nu{}^a A_\alpha + b_2 f_{\mu,\nu} f_\alpha{}^a B^a + b_3 \Omega_{\mu,\nu} A_\alpha \varphi + b_4 f_{\mu,\nu} B_\alpha \varphi] \\
&\quad + b_5 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a A_\nu \pi^b + b_6 \varphi (\pi A)
\end{aligned}
$$

η^a **transformations:**

$$
\begin{aligned}
&\varepsilon^{\mu\nu\alpha}[b_1 D_\mu f_\nu{}^a \eta^a A_\alpha - (b_1 + 2m\alpha_1) f_\mu{}^a \eta^a D_\nu A_\alpha + (b_3 - 2ma_4)\eta_\mu D_\nu A_\alpha \varphi - b_3 \eta_\mu A_\nu D_\alpha \varphi] + \\
&+ (b_1 - 4m\kappa_3) \left\{ {}^{\mu\nu}_{ab} \right\} \Omega_\mu{}^a A_\nu \eta^b + 2b_2 \eta^\mu f_\mu{}^a B^a + (b_2 + m\alpha_1) \left\{ {}^{\mu\nu}_{ab} \right\} f_{\mu,\nu} B^a \eta^b + \\
&+ 2m\alpha_1 f(B\eta) + (2b_4 - 4ma_3)\varphi(B\eta) + 2m\kappa_3 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a B_\nu \eta^b + (b_5 + 2M\alpha_2)\varepsilon^{\mu\nu\alpha} \pi_\nu \eta_\alpha A_\mu
\end{aligned}
$$

The most general form of corrections would be:

$$
\delta \Omega_\mu{}^a = \beta_1 A_\mu \eta^a, \qquad \delta B_\mu = \beta_2 f_\mu{}^a \eta^a + \beta_3 \varphi \eta_\mu
$$

but here we use remaining field redefinition (18) and put $\beta_3 = 0$. Then all variations can be cancelled provided

$$
\alpha_1 = -\kappa_3, \qquad \beta_1 = 2m\kappa_3, \qquad \beta_2 = 0, \qquad a_4 = 0
$$

$$
b_1 = 2m\kappa_3, \qquad b_2 = m\kappa_3, \qquad b_3 = 0, \qquad b_4 = 2ma_3, \qquad b_5 = -2M\alpha_2
$$

ξ^a **transformations:**

$$
\varepsilon^{\mu\nu\alpha}[-b_1 D_\mu \Omega_\nu{}^a A_\alpha \xi^a - b_2 D_\mu f_{\nu,\alpha}(B\xi) + b_1 \Omega_\mu{}^a \xi^a D_\nu A_\alpha] + \left\{ {}^{\mu\nu}_{ab} \right\} [2m\alpha_1 \Omega_{\mu,\nu} B^a \xi^b + 2m\kappa_3 \Omega_\mu{}^a B_\nu \xi^b]
$$

$$
(b_4 + ma_4)\varepsilon^{\mu\nu\alpha} \xi_\mu D_\nu B_\alpha \varphi - b_5 \left\{ {}^{\mu\nu}_{ab} \right\} \xi^a \pi^b D_\mu A_\nu - (b_4 + ma_4)\varepsilon^{\mu\nu\alpha} \xi_\mu B_\nu D_\alpha \varphi - 2M\alpha_1 \varepsilon^{\mu\nu\alpha} \pi_\mu B_\nu \xi_\alpha
$$

Thus we need the following corrections:

$$
\begin{aligned}
\delta \Omega_\mu{}^a &= -m\kappa_3 e_\mu{}^a (B\xi), \qquad \delta f_\mu{}^a = -2m\kappa_3 A_\mu \xi^a \\
\delta B_\mu &= 2n\kappa_3 \Omega_\mu{}^a \xi^a - 2M\alpha_2 \varepsilon_\mu{}^{ab} \pi^a \xi^b, \qquad \delta A_\mu = 2ma_3 \varphi \xi_\mu \\
\delta \pi^a &= -2ma_3 \varepsilon^{abc} B^b \xi^c
\end{aligned}
$$

In this, cancellation of such variations requires

$$
2M(\alpha_2 - \alpha_1) + 2ma_3 = 0
$$

ξ **transformations:**

$$
\begin{aligned}
&-2m\kappa_3 \varepsilon^{\mu\nu\alpha}[D_\mu f_\nu{}^a \Omega_\alpha{}^a \xi - f_\mu{}^a D_\nu \Omega_\alpha{}^a \xi] + 2m\kappa_3 \left\{ {}^{\mu\nu}_{ab} \right\} \Omega_\mu{}^a \Omega_\nu{}^b \xi - \\
&-(3m\alpha_1 + 2Ma_3) B^a B^a \xi + (2m\alpha_1 - 2Ma_4)\varepsilon^{\mu\nu\alpha} B_\mu D_\nu A_\alpha \xi - \\
&-(b_6 + 4m\alpha_2)\pi^\mu D_\mu \varphi \xi - b_6 \varphi (D\pi)\xi + 3m\alpha_2 \pi^a \pi^a \xi
\end{aligned}
$$

This time we introduce corrections of the form:

$$
\begin{aligned}
\delta \Omega_\mu{}^a &= -2m\kappa_3 \Omega_\mu{}^a \xi, \qquad \delta f_\mu{}^a = 2m\kappa_3 f_\mu{}^a \xi \\
\delta B^a &= (2m\alpha_1 - 2Ma_4) B^a \xi \\
\delta \pi^a &= 3m\alpha_2 \pi^a \xi, \qquad \delta \varphi = m\alpha_2 \varphi \xi
\end{aligned}
$$

In this, all such variations cancel provided

$$2M(a_3 + a_4) = -m\alpha_1, \qquad b_6 = -m\alpha_2$$

Note that combining the results from ξ^a and ξ transformations we obtain:

$$a_3 = -\frac{m\alpha_1}{2M}, \qquad \alpha_2 = (1 + \frac{m^2}{2M^2})\alpha_1 \tag{21}$$

Collecting all pieces together we obtain:

$$\begin{aligned} \mathcal{L}_{11} &= m\varepsilon^{\mu\nu\alpha}[2\kappa_3 f_\mu{}^a \Omega_\nu{}^a A_\alpha - \alpha_1 f_{\mu,\nu} f_\alpha{}^a B^a + 2a_3 f_{\mu,\nu} B_\alpha \varphi] - \\ &\quad -2M\alpha_2 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a A_\nu \pi^b - m\alpha_2 \varphi(\pi A) \end{aligned} \tag{22}$$

$$\begin{aligned} \delta_{11}\Omega_\mu{}^a &= 2m\kappa_3 A_\mu \eta^a - m\kappa_3 e_\mu{}^a (B\xi) - 2m\kappa_3 \Omega_\mu{}^a \xi, \qquad \delta_{11} f_\mu{}^a = -2m\kappa_3 A_\mu \xi^a + 2m\kappa_3 f_\mu{}^a \xi \\ \delta_{11} B_\mu &= 2m\kappa_3 \Omega_\mu{}^a \xi^a - 2M\alpha_2 \varepsilon_\mu{}^{ab} \pi^b \xi^c + 2m\alpha_1 B_\mu \xi, \qquad \delta_{11} A_\mu = 2ma_3 \varphi \xi_\mu \\ \delta_{11}\pi^a &= -2ma_3 \varepsilon^{abc} B^b \xi^c + 3m\alpha_2 \pi^a \xi, \qquad \delta_{11}\varphi = m\alpha_2 \varphi \xi \end{aligned} \tag{23}$$

Now let us turn to the variations of order m^2. Additional terms to Lagrangian look as follows:

$$\mathcal{L}_{12} = c_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} f_\mu{}^a f_\nu{}^b f_\alpha{}^c + c_2 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b \varphi + c_3 f \varphi^2 + c_4 \varphi^3 \tag{24}$$

η^a **transformations:**

$$\varepsilon^{\mu\nu\alpha}[(6c_1 + 2m^2\alpha_1 - 2M^2\kappa_3) f_{\mu,\nu} f_\alpha{}^a \eta^a + (2c_2 - 4m^2 a_3 - 4mM\kappa_3) f_{\mu,\nu} \eta_\alpha \varphi]$$

This gives us:

$$c_1 = \frac{(M^2 + m^2)}{3}, \qquad c_2 = 2m^2 a_3 + 2mM\kappa_3 = -2mM\alpha_2$$

ξ^a **transformations:**

$$\begin{aligned} &-6c_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} D_\mu f_\nu{}^a f_\alpha{}^b \xi^c + 6c_1 \varepsilon^{\mu\nu\alpha}[\Omega_\mu{}^a \xi^a f_{\nu,\alpha} + f_\mu{}^a \Omega_\nu{}^a \xi_\alpha] + \\ &+2c_2 \left\{ {}^{\mu\nu}_{ab} \right\} D_\mu f_\nu{}^a \xi^b \varphi - 2c_2 \varepsilon^{\mu\nu\alpha} \Omega_{\mu,\nu} \xi_\alpha \varphi - \\ &-2(c_2 + mM\alpha_2) \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a D_\nu \varphi \xi^b - 2mM\alpha_2 \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a \xi_\nu \pi^b - 2mM\alpha_2 \left\{ {}^{\mu\nu}_{ab} \right\} \pi^a \xi^b f_{\mu,\nu} - \\ &-2c_3 \xi^\mu \varphi D_\mu \varphi - 7m^2\alpha_2 \varphi(\pi\xi) + 4mMa_3 \varphi(\pi\xi) \end{aligned}$$

Here we need the following corrections:

$$\begin{aligned} \delta_{12}\Omega_\mu{}^a &= -6c_1 \varepsilon^{abc} f_\mu{}^b \xi^c + 2c_2 \varepsilon_\mu{}^{ab} \varphi \xi^b \\ \delta_{12}\pi^a &= 2mM\alpha_2(\xi^\mu f_\mu{}^a - f\xi^a) + 2c_3 \varphi \xi^a \end{aligned} \tag{25}$$

Then all variations cancel provided

$$2c_3 + 7m^2\alpha_2 - 2m^2\kappa_3 = 0$$

In this, all ξ variations also cancel.

We still have variations of order m^3. As we have checked variations for ξ^a transformations cancel, while for ξ transformations we get

$$6(mc_3 - Mc_4 + m^3\alpha_2)\varphi^2\xi$$

This gives us an expression for last unknown coefficient c_4:

$$Mc_4 = m^3(\kappa_3 - \frac{5}{2}\alpha_2)$$

Thus we have complete set of cubic vertices (19), (22), (24) and corresponding corrections to gauge transformations (20), (23), (25). Note that as we will see in the next subsection for usual gravitational interactions we must have $\alpha_1 = \alpha_2 = -2\kappa_3$, but for massive spin 2 self-interaction we obtained[5]

$$\alpha_1 = -\kappa_3, \qquad \alpha_2 = -(1 + \frac{m^2}{2M^2})\kappa_3$$

This result is a consequence of spontaneously broken symmetries with Stueckelberg fields providing their non-linear realization. From the last relation above it follows that it is impossible to take partially massless limit $M \to 0$ in the interacting theory.

Contrary to the massless case due to the presence of spin 1 and spin 0 fields there exist (and necessarily must be present) quartic and higher vertices so that the results obtained is not complete theory yet. We will return to this point in Subsection 2.4.

2.3 Gravitational interaction

In this subsection we consider gravitational interactions for massive spin 2 particles, i.e. cross-interaction for massless and massive ones. We have explicitly checked that the only solution possible exactly corresponds to standard minimal gravitational interactions, i.e. can be obtained by the usual rule where background frame $e_\mu{}^a$ is replaced by dynamical one $h_\mu{}^a$ while AdS covariant derivatives are replaced by fully Lorentz covariant ones. Thus we will not give details of calculations here (they are similar to those in previous subsection) and just present the final results. Here complete set of cubic vertices also consists of three parts:

$$\mathcal{L}_1 = \mathcal{L}_{10} + \mathcal{L}_{11} + \mathcal{L}_{12} \tag{26}$$

$$\begin{aligned}
\mathcal{L}_{10} &= \kappa_2 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [h_\mu{}^a\Omega_\nu{}^b\Omega_\alpha{}^c + 2f_\mu{}^a\omega_\nu{}^b\Omega_\alpha{}^c] + \kappa_2 hB^aB^a - 2\kappa_2\varepsilon^{\mu\nu\alpha}h_\mu{}^aB^aD_\nu A_\alpha - \\
&\quad - \kappa_2 h\pi^a\pi^a + 2\kappa_2 \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a D_\nu\varphi\pi^b \\
\mathcal{L}_{11} &= 2m\kappa_2\varepsilon^{\mu\nu\alpha}[2h_\mu{}^a\Omega_\nu{}^aA_\alpha + h_\mu{}^aB^af_{\nu,\alpha} - B_\mu h_\nu{}^af_\alpha{}^a] + 4M\kappa_2 \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^aA_\nu\pi^b \\
\mathcal{L}_{12} &= M^2\kappa_2 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^af_\nu{}^bf_\alpha{}^c + 4mM\kappa_2 \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^af_\nu{}^b\varphi + 6m^2\kappa_2 h\varphi^2
\end{aligned}$$

while appropriate corrections to gauge transformations look as follows:

$$\delta_1 = \delta_{10} + \delta_{11} + \delta_{12} \tag{27}$$

[5]In agreement with the general results from metric-like formalism [27].

$$
\begin{aligned}
\delta_{10}\omega_\mu{}^a &= -2\sigma\kappa_2\varepsilon^{abc}\Omega_\mu{}^b\eta^c, \qquad \delta_{10}h_\mu{}^a = -2\sigma\kappa_2\varepsilon^{abc}[f_\mu{}^b\eta^c + \Omega_\mu{}^b\xi^c] \\
\delta_{10}\Omega_\mu{}^a &= -2\kappa_2\varepsilon^{abc}[\Omega_\mu{}^b\hat{\eta}^c + \omega_\mu{}^b\eta^c] \\
\delta_{10}f_\mu{}^a &= -2\kappa_2\varepsilon^{abc}[f_\mu{}^b\hat{\eta}^c + \Omega_\mu{}^b\hat{\xi}^c + h_\mu{}^b\eta^c + \omega_\mu{}^b\xi^c] \\
\delta_{10}B^a &= -2\kappa_2\varepsilon^{abc}B^b\hat{\eta}^c + 2\kappa_2\hat{\xi}^b D^a B^b, \qquad \delta_{10}A_\mu = 2\kappa_2\varepsilon_\mu{}^{ab}B^a\hat{\xi}^b \\
\delta_{10}\pi^a &= -2\kappa_2\varepsilon^{abc}\pi^b\hat{\eta}^c + 2\kappa_2(\hat{\xi}^\mu D_\mu\pi^a - \hat{\xi}^a(D\pi)), \qquad \delta_{10}\varphi = 2\kappa_2(\pi\hat{\xi})
\end{aligned}
$$

$$
\begin{aligned}
\delta_{11}\omega_\mu{}^a &= 2m\sigma\kappa_2(2A_\mu\eta^a - B_\mu\xi^a - 2\Omega_\mu{}^a\xi) \\
\delta_{11}\Omega_\mu{}^a &= 2m\kappa_2(B_\mu\hat{\xi}^a - e_\mu{}^a(B\hat{\xi})) \\
\delta_{11}f_\mu{}^a &= 4m\kappa_2(-A_\mu\hat{\xi}^a + h_\mu{}^a\xi) \\
\delta_{11}B_\mu &= 4\kappa_2(m\Omega_\mu{}^a\hat{\xi}^a + M\varepsilon_\mu{}^{ab}\pi^a\hat{\xi}^b) \\
\delta_{11}A_\mu &= 2m\kappa_2(-f_\mu{}^a\hat{\xi}^a + h_\mu{}^a\xi^a)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{12}\omega_\mu{}^a &= 2M\sigma\kappa_2(-M\varepsilon^{abc}f_\mu{}^b\xi^c + 2m\varepsilon_\mu{}^{ab}\varphi\xi^b) \\
\delta_{12}\Omega_\mu{}^a &= 2M\kappa_2(-M\varepsilon^{abc}f_\mu{}^b\hat{\xi}^c + 2m\varepsilon_\mu{}^{ab}\varphi\hat{\xi}^b - M\varepsilon^{abc}h_\mu{}^b\xi^c) \\
\delta_{12}\pi^a &= 4m\kappa_2[-M(\hat{\xi}^\mu f_\mu{}^a - f\hat{\xi}^a) + 3m\varphi\hat{\xi}^a]
\end{aligned}
$$

Note that in this case nothing prevent us from taking partially massless limit $M \to 0$ so that at least in the linear approximation it is possible to obtain gravitational interactions for partially massless spin 2 particles.

2.4 Beyond linear approximation

As we have already mentioned, due to the presence of spin 1 and spin 0 components there must be quartic (and even higher) vertices. So, contrary to the massless case, the linear approximation considered in the two previous subsections is not the end of the story. But all the terms that includes spin 2 fields only have already been fixed. Thus, if we require that the model we are looking for does admit non-singular massless limit, we may try to put some restriction on the parameters. Recall that in the massless case we have

$$
\mathcal{L}_{10} = \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [\kappa_0 h_\mu{}^a\omega_\nu{}^b\omega_\alpha{}^c + \kappa_2 h_\mu{}^a\Omega_\nu{}^b\Omega_\alpha{}^c + 2\kappa_2 f_\mu{}^a\omega_\nu{}^b\Omega_\alpha{}^c + \kappa_3 f_\mu{}^a\Omega_\nu{}^b\Omega_\alpha{}^c]
$$

and all quadratic variations for $\hat{\eta}^a$, $\hat{\xi}^a$, η^a and ξ^a transformations cancelled provided $\kappa_2 = \sigma\kappa_0$ with arbitrary κ_3. But in the massive case we have additional symmetry:

$$
\begin{aligned}
\delta\omega_\mu{}^a &= -4m\sigma\kappa_2\Omega_\mu{}^a\xi, \qquad \delta\Omega_\mu{}^a = -2m\kappa_3\Omega_\mu{}^a\xi \\
\delta f_\mu{}^a &= 4m\kappa_2 h_\mu{}^a\xi + 2m\kappa_3 f_\mu{}^a\xi
\end{aligned}
$$

where we collected all terms form both previous subsections. It was not evident from the very beginning but it turns out that cancellation for quadratic ξ variations is indeed possible provided the following relation holds:

$$
4\sigma{\kappa_0}^2 + {\kappa_3}^2 = 0 \quad \Longrightarrow \quad \sigma = -1
$$

As can easily be seen this one relation gives us two important results. First, we get a relation between two previously independent coupling constants. Second, this solution exists for $\sigma = -1$ only when massless graviton is a ghost exactly as in the so called "New massive gravity".

Conclusion

Thus we have seen that constructive approach based on the frame-like gauge invariant description of massive spin 2 particles does allow one to systematically investigate possible consistent ghost-free models though due to large number of fields involved this requires much more work than in the massless case. It is evident that such approach admits straightforward generalization to higher spins. In this in three dimensional case we will have to nice features making investigations simpler — there no any quartic vertices for any spins $s \geq 2$ and also there are no so called extra fields and thus there is no need in higher derivatives. So we may hope to gain some useful experience for work with massive higher spin fields. Also it is worth noting that such approach can be applied to higher dimensional theories as well, in this, first of all it would be interesting to understand peculiar features of massive gravity and bigravity in $d = 4$. Work is in progress in both directions.

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A On cubic vertex with two spin 2 and one spin 0

The most general form for such cubic vertex with two derivatives:

$$\begin{aligned}
\mathcal{L}_1 &= a_1 \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a \omega_\nu{}^b \varphi + a_2 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b D_\alpha \pi^c + \\
&+ \varepsilon^{\mu\nu\alpha} [a_3 \omega_\mu{}^a h_\nu{}^a \pi_\alpha + a_4 \omega_{\mu,\nu} h_\alpha{}^a \pi^a + a_5 \omega_\alpha{}^a \pi^a h_{\nu,\alpha} + a_6 \omega_\mu{}^a D_\nu h_\alpha{}^a \varphi + a_7 \omega_\mu{}^a h_\nu{}^a D_\alpha \varphi]
\end{aligned}$$

In this, there exist three possible field redefinitions:

$$\omega_\mu{}^a \Rightarrow \omega_\mu{}^a + \rho_1 \varphi \omega_\mu{}^a + \rho_2 \varepsilon^{abc} h_\mu{}^b \pi^c, \qquad h_\mu{}^a \Rightarrow h_\mu{}^a + \rho_3 \varphi h_\mu{}^a$$

Let us consider variations under $\hat{\eta}^a$ transformations:

$$\begin{aligned}
\delta_{\hat{\eta}} \mathcal{L}_1 &= (2a_1 + a_6) \left\{ {}^{\mu\nu}_{ab} \right\} D_\mu \omega_\nu{}^a \hat{\eta}^b \varphi + \\
&+ \varepsilon^{\mu\nu\alpha} [-a_3 \hat{\eta}^a D_\mu h_\nu{}^a \pi_\alpha + a_4 \hat{\eta}_\mu D_\nu h_\alpha{}^a \pi^a - a_5 (\pi\hat{\eta}) D_\mu h_{\nu,\alpha} - (a_6 + a_7) \hat{\eta}^a D_\mu h_\nu{}^a D_\alpha \varphi] + \\
&+ \varepsilon^{\mu\nu\alpha} [(2a_2 + a_3) h_\mu{}^a \hat{\eta}^a D_\nu \pi_\alpha + (2a_2 - a_5) h_{\mu,\nu} \hat{\eta}^a D_\alpha \pi^a - a_4 \hat{\eta}_\mu h_\nu{}^a D_\alpha \pi^a] + \\
&+ \left\{ {}^{\mu\nu}_{ab} \right\} [-a_3 \omega_\mu{}^a \pi_\nu \hat{\eta}^b + a_4 \omega_{\mu,\nu} \pi^a \hat{\eta}^b] + a_5 \hat{\eta}^\mu \omega_\mu{}^a \pi^a - (2a_1 + a_6 + a_7) \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a D_\nu \varphi \hat{\eta}^b
\end{aligned}$$

From the third line it follows that $a_3 = -2a_2$, $a_4 = 0$, $a_5 = 2a_2$. But in this case terms with coefficients $a_{2,3,5}$ can be removed by redefinition with parameter ρ_2. This leaves us with

$$(2a_1 + a_6) \left\{ {}^{\mu\nu}_{ab} \right\} D_\mu \omega_\nu{}^a \hat{\eta}^b \varphi - (a_6 + a_7) \varepsilon^{\mu\nu\alpha} D_\mu f_\nu{}^a \hat{\eta}^a D_\alpha \varphi - (2a_1 + a_6 + a_7) \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a \hat{\eta}^b D_\nu \varphi$$

As usual to compensate them we introduce corrections to gauge transformations:

$$\delta h_\mu{}^a = \alpha_1 \varepsilon_\mu{}^{ab} \hat{\eta}^b \varphi, \qquad \delta \omega_\mu{}^a = \alpha_2 D_\mu \varphi \hat{\eta}^a$$

They give additional contribution:

$$-\alpha_1 \left\{ {}^{\mu\nu}_{ab} \right\} D_\mu \omega_\nu{}^a \hat{\eta}^b \varphi - \alpha_2 \varepsilon^{\mu\nu\alpha} D_\mu f_\nu{}^a \hat{\eta}^a D_\alpha \varphi + \alpha_2 \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a \hat{\eta}^b D_\nu \varphi$$

Hence $a_1 + a_6 + a_7 = 0$ and all remaining terms can be removed by field redefinitions with parameters ρ_1 and ρ_3.

B On cubic vertex with two massless spin 2 and one massive one

As it has been explained in the Subsection 2.2 we will look for cubic vertices and appropriate corrections to gauge transformations in the form

$$\mathcal{L}_1 = \mathcal{L}_{10} + \mathcal{L}_{11} + \mathcal{L}_{12}, \qquad \delta_1 = \delta_{10} + \delta_{11} + \delta_{12}$$

Moreover, it turns out that to see that such vertex does not exist it is enough to consider gauge transformations for massive spin 2 field only. Taking into account results of Appendix A, the most general possibility for $\mathcal{L}_{10}$ is:

$$\mathcal{L}_1 = \kappa_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} [f_\mu{}^a \omega_\nu{}^b \omega_\alpha{}^c + 2h_\mu{}^a \omega_\nu{}^b \Omega_\alpha{}^c]$$

while corresponding corrections to gauge transformations were given by formula (6) in Subsection 1.2.

Let us turn to variations of order m. The most general additional terms for the Lagrangian have the form:

$$\mathcal{L}_{11} = \varepsilon^{\mu\nu\alpha}[b_1 h_\mu{}^a \omega_\nu{}^a A_\alpha + b_2 h_\mu{}^a B^a h_{\nu,\alpha}]$$

In this order there are no variations for η^a and ξ^a transformations while for ξ transformations we obtain:

$$\varepsilon^{\mu\nu\alpha}[-b_1 D_\mu h_\nu{}^a \omega_\alpha{}^a \xi + b_1 h_\mu{}^a D_\nu \omega_\alpha{}^a \xi] + 2ma_1 \left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^a \omega_\nu{}^b \xi$$

This variations can be compensated by the following corrections:

$$\delta_1 \omega_\mu{}^a = -b_1 \omega_\mu{}^a \xi, \qquad \delta_1 h_\mu{}^a = b_1 h_\mu{}^a \xi$$

provided $b_1 = 2ma_1$.

We proceed with the variation of order m^2 and introduce the last part of the Lagrangian

$$\mathcal{L}_{12} = c_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b f_\alpha{}^c + c_2 \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \varphi$$

Variations under ξ^a transformations

$$\begin{aligned} &-2c_1 \left\{ {}^{\mu\nu\alpha}_{abc} \right\} D_\mu h_\nu{}^a h_\alpha{}^b \xi^c + 2mb_1 \varepsilon^{\mu\nu\alpha} h_\mu{}^a \omega_\nu{}^a \xi_\alpha + \\ &+2\kappa_1 M^2 \varepsilon^{\mu\nu\alpha}[h_\mu{}^a \omega_{\nu,\alpha} + h_{\mu,\nu}\omega_\alpha{}^a]\xi^a - 2\kappa_1 \lambda^2 \varepsilon^{\mu\nu\alpha}[h_\mu{}^a \omega_{\nu,\alpha}\xi^a - h_\mu{}^a \omega_\nu{}^a \xi_\alpha] \end{aligned}$$

require corrections

$$\delta \omega_\mu{}^a = -2c_1 \varepsilon^{abc} h_\mu{}^b \xi^c$$

and we obtain:

$$\varepsilon^{\mu\nu\alpha}[\kappa_2(M^2 - \lambda^2) h_\mu{}^a \omega_{\nu,\alpha}\xi^a + (\kappa_2 M^2 - c_1) h_{\mu,\nu}\omega_\alpha{}^a \xi^a + (\kappa_2 m^2 + \kappa_2 \lambda^2 - c_1) h_\mu{}^a \omega_\nu{}^a \xi_\alpha = 0$$

It is easy to see that solution is possible for $m = 0$ only.

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